Linearized conformal gravity

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Abstract

We examine the linearization of Weyl conformal gravity, showing that the only solutions are also solutions to linearized general relativity.

1 The Bach equation

The Weyl gravity action is

$$S = \int C^{\alpha\mu\beta\nu} C_{\alpha\mu\beta\nu} \sqrt{-g} d^4 x$$

It leads, in vacuum, to the Bach equation,

$$D_\mu D_\nu C^{\alpha\mu\beta\nu} - \frac{1}{2} R_{\mu\nu} C^{\alpha\mu\beta\nu} = 0$$

It is generally easier to compute from the alternate form of the Bach equation,

$$\begin{aligned} W_{\alpha\beta} &= -\frac{1}{3} D_\alpha D_\beta R + D_\mu D^\mu R_{\alpha\beta} + \frac{1}{6}\left(R^2 - D_\mu D^\mu R - 3R_{\mu\nu}R^{\mu\nu}\right) g_{\alpha\beta} \\ &\quad + 2R^{\mu\nu} R_{\alpha\mu\beta\nu} - \frac{2}{3} R R_{\alpha\beta} \end{aligned} \tag{1}$$

which is found by expanding the Weyl curvatures in the action in terms of the Riemann curvature, then using the Gauss-Bonnet expression for the Euler character to eliminate all but the Ricci and Ricci scalar terms.

2 Linearized gravity

The linearization of general relativity is well established, either by perturbing around flat space or about a fixed background. We consider first a flat background, writing

$$\begin{aligned} g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta} \\ g^{\alpha\beta} &= \eta^{\alpha\beta} - h^{\alpha\beta} \end{aligned}$$

where the components of $h_{\alpha\beta}$ are regarded as perturbatively small and $h^{\alpha\beta} \equiv \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$. Then at first order, $h_{\alpha\beta}$ acts as a field on flat spacetime. The connection is

$$\Gamma^{\alpha}_{\ \beta\mu} = \frac{1}{2}\eta^{\alpha\nu}\left(h_{\nu\beta,\mu} + h_{\nu\mu,\beta} - h_{\beta\mu,\nu}\right)$$

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The curvature becomes

$$\begin{aligned} R^{\alpha}_{\ \beta\mu\nu} &= \Gamma^{\alpha}_{\ \beta\mu,\nu} - \Gamma^{\alpha}_{\ \beta\nu,\mu} - \Gamma^{\alpha}_{\ \rho\mu}\Gamma^{\rho}_{\ \beta\nu} + \Gamma^{\alpha}_{\ \rho\nu}\Gamma^{\rho}_{\ \beta\mu} \\ &\approx \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\sigma\beta,\mu\nu} + h_{\sigma\mu,\beta\nu} - h_{\beta\mu,\sigma\nu}\right) - \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\sigma\beta,\mu\nu} + h_{\sigma\nu,\beta\mu} - h_{\beta\nu,\sigma\mu}\right) \\ &= \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\sigma\mu,\beta\nu} - h_{\beta\mu,\sigma\nu} - h_{\sigma\nu,\beta\mu} + h_{\beta\nu,\sigma\mu}\right) \end{aligned}$$

It is useful to define the trace-reversed variable,

$$\begin{aligned} \bar{h}_{\alpha\beta} &\equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \\ h &\equiv \eta^{\mu\nu}h_{\mu\nu} \end{aligned}$$

The Ricci tensor becomes

$$\begin{aligned} R^{\alpha}_{\ \beta\alpha\nu} &= \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\sigma\alpha,\beta\nu} - h_{\beta\alpha,\sigma\nu} - h_{\sigma\nu,\beta\alpha} + h_{\beta\nu,\sigma\alpha}\right) \\ &= \frac{1}{2}\left(h_{,\beta\nu} - 2h^{\alpha}_{\ \nu,\beta\alpha} + \Box h_{\beta\nu}\right) \\ &= \frac{1}{2}\left(h_{,\beta\nu} + \frac{1}{2}\eta_{\beta\nu}\Box h - h^{\alpha}_{\ \beta,\alpha\nu} - h^{\alpha}_{\ \nu,\alpha\beta} + \Box\left(h_{\beta\nu} - \frac{1}{2}\eta_{\beta\nu}h\right)\right) \end{aligned}$$

The infinitesimal coordinate freedom of the metric,

$$g_{\alpha\beta} \to g_{\alpha\beta} + k_{\alpha;\beta} + k_{\beta;\alpha}$$

allows us to change $h_{\alpha\beta}$,

$$\tilde{h}_{\alpha\beta} = h_{\alpha\beta} + k_{\alpha,\beta} + k_{\beta,\alpha}$$

to make it divergence free,

$$\begin{aligned} 0 &= \tilde{h}^{\alpha\beta}_{\ \ \ ,\beta} \\ &= h^{\alpha\beta}_{\ \ \ ,\beta} + \Box k^{\alpha} + k^{\beta,\alpha}_{\ \ \ \ ,\beta} \end{aligned}$$

An additional transformation may make it traceless. The fully transverse-traceless mode for $h_{\alpha\beta}$ allows us to write it as

$$h_{\alpha\beta} = \begin{pmatrix} 0 & & & \\ & h_{+} & h_{-} & \\ & h_{-} & -h_{+} & \\ & & & 0 \end{pmatrix}$$

Assuming the transverse traceless conditions hold for $h_{\alpha\beta}$, the Riemann tensor, Ricci tensor and Ricci scalar reduce to

$$\begin{aligned} R^{\alpha}_{\ \beta\mu\nu} &= \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\sigma\mu,\beta\nu} - h_{\beta\mu,\sigma\nu} - h_{\sigma\nu,\beta\mu} + h_{\beta\nu,\sigma\mu}\right) \\ R_{\alpha\beta} &= \Box h_{\alpha\beta} \\ R &= 0 \end{aligned}$$

3 Substituting into the Bach tensor

Now substitute into

$$\begin{aligned} W_{\alpha\beta} &= -\frac{1}{3}D_{\alpha}D_{\beta}R + D_{\mu}D^{\mu}R_{\alpha\beta} + \frac{1}{6}\left(R^2 - D_{\mu}D^{\mu}R - 3R_{\mu\nu}R^{\mu\nu}\right)g_{\alpha\beta} \\ &\quad +2R^{\mu\nu}R_{\alpha\mu\beta\nu} - \frac{2}{3}RR_{\alpha\beta} \\ &= D_{\mu}D^{\mu}R_{\alpha\beta} \\ &= \Box\Box h_{\alpha\beta} \end{aligned}$$

We seek vacuum solutions for $W_{\alpha\beta} = 0$.

Setting

$$\Box\Box h_{\alpha\beta} = 0$$

or equivalently, the pair

$$\begin{aligned} \Box\Box h_{+} &= 0 \\ \Box\Box h_{-} &= 0 \end{aligned}$$

we may solve for each component of $\Box h_{\pm}$ in the usual manner, by Fourier expansion,

$$\begin{aligned} \Box h_{\pm} &= \int d^4k \left(A_{\pm}\left(k^{\mu}\right) e^{ik_{\beta}x^{\nu}} \delta\left(k^{\beta}k_{\beta}\right)\right) \\ &= \int d^4k \left(A_{\pm}\left(k^{\mu}\right) e^{ik_{\nu}x^{\nu}} \delta\left(\omega^2 - \mathbf{k}\cdot\mathbf{k}\right)\right) \\ &= \int \frac{d^3k}{2\omega} \left(A_{\pm}\left(k^{i}\right) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + B_{\pm}\left(k^{i}\right) e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}\right) \end{aligned}$$

where

$$\begin{aligned} \omega &= \omega\left(\mathbf{k}\right) \\ &= |\mathbf{k}| \end{aligned}$$

This solution to the wave equation now acts as source for the same wave equation. To solve, we need the Green's function. Choosing vanishing boundary conditions at infinity this is straightforward to derive. We need

$$\Box G\left(x^{\alpha}, x_0^{\alpha}\right) = -4\pi\delta^4\left(x^{\alpha} - x_0^{\alpha}\right)$$

Taking the Fourier transform,

$$\begin{aligned} \frac{1}{4\pi^2}\Box\int d^4k\, G\left(k^{\nu}\right) e^{ik_{\alpha}\left(x^{\alpha}-x_0^{\alpha}\right)} &= -\frac{4\pi}{16\pi^4}\int d^4k\, e^{ik_{\alpha}\left(x^{\alpha}-x_0^{\alpha}\right)} \\ \frac{1}{4\pi^2}\int d^4k \left(k_{\beta}k^{\beta}G\left(k^{\nu}\right) + \frac{1}{\pi}\right) e^{ik_{\alpha}\left(x^{\alpha}-x_0^{\alpha}\right)} &= 0 \\ G\left(k^{\nu}\right) &= -\frac{1}{\pi k_{\beta}k^{\beta}} \end{aligned}$$

Therefore,

$$\begin{aligned} G\left(x^{\alpha}, x_0^{\alpha}\right) &= \frac{1}{4\pi^2}\int d^4k\, G\left(k^{\nu}\right) e^{ik_{\alpha}\left(x^{\alpha}-x_0^{\alpha}\right)} \\ &= -\frac{1}{4\pi^3}\int d^4k\, \frac{1}{k_{\beta}k^{\beta}} e^{ik_{\alpha}\left(x^{\alpha}-x_0^{\alpha}\right)} \\ &= -\frac{1}{4\pi^3}\int d^3k\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}_0)}\int d\omega\, \frac{e^{-i\omega(t-t_0)}}{\omega^2 - \mathbf{k}\cdot\mathbf{k}} \end{aligned}$$

Infinitesimally offsetting the poles by replacing $\omega \to \omega + i\varepsilon$, we complete contours by half circles at infinity.

$$\begin{aligned} \int d\omega\, \frac{e^{-i\omega t}}{\omega^2 - \mathbf{k}\cdot\mathbf{k}} &= \int d\omega\, \frac{e^{-i\omega t}}{\left(\omega + \sqrt{\mathbf{k}\cdot\mathbf{k}}\right)\left(\omega - \sqrt{\mathbf{k}\cdot\mathbf{k}}\right)} \\ &\to \int d\omega\, \frac{e^{-i\omega t + \varepsilon t}}{\left(\omega + \sqrt{\mathbf{k}\cdot\mathbf{k}} + i\varepsilon\right)\left(\omega - \sqrt{\mathbf{k}\cdot\mathbf{k}} + i\varepsilon\right)} \end{aligned}$$

For $t<0$, we complete the contour in the upper half plane and there are no poles enclosed. For $t>0$, we complete the contour with a half circle at infinity in the lower half plane and there are two poles. Letting $\varepsilon \to 0$, the integral becomes

$$\begin{aligned}
\int d\omega \, \frac{e^{-i\omega t}}{\omega^2 - \mathbf{k}\cdot\mathbf{k}} &= 2\pi i \Theta(t) \left(\frac{e^{i\omega(k)t}}{-2\omega(k)} + \frac{e^{-i\omega(k)t}}{2\omega(k)} \right) \\
&= \frac{2\pi}{\omega} \Theta(t) \left(\frac{e^{i\omega(k)t} - e^{-i\omega(k)t}}{2i} \right) \\
&= \frac{2\pi}{\omega} \Theta(t) \sin \omega(k) t
\end{aligned}$$

where $\omega(k) = +\sqrt{\mathbf{k}\cdot\mathbf{k}} \equiv k$. We are left with

$$\begin{aligned}
G(x^\alpha, x_0^\alpha) &= -\frac{1}{4\pi^2 i} \Theta(t-t_0) \int \frac{d^3k}{\omega(k)} \, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0)} \left(e^{i\omega(k)(t-t_0)} - e^{-i\omega(k)(t-t_0)} \right) \\
&= \frac{1}{4\pi^2 i} \Theta(t-t_0) \int \frac{d^3k}{\omega(k)} \left(e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0) - \omega(t-t_0))} - e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0) + \omega(t-t_0))} \right)
\end{aligned}$$

For the remaining integrals,

$$\begin{aligned}
\int d^3k \, \frac{e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0) - \omega(t-t_0))}}{\omega(k)} &= \int k^2 dk d\varphi d(\cos\theta) \, \frac{e^{ik(|\mathbf{x}-\mathbf{x}_0|\cos\theta - (t-t_0))}}{k} \\
&= 2\pi \int k^2 dk d(\cos\theta) \, \frac{e^{ik(|\mathbf{x}-\mathbf{x}_0|\cos\theta - (t-t_0))}}{k} \\
&= 2\pi \int k^2 dk \, \frac{\left(e^{ik|\mathbf{x}-\mathbf{x}_0|} - e^{-ik|\mathbf{x}-\mathbf{x}_0|} \right) e^{-ik(t-t_0)}}{ik^2 |\mathbf{x}-\mathbf{x}_0|} \\
&= \frac{2\pi}{i|\mathbf{x}-\mathbf{x}_0|} \int dk \, \left(e^{ik(|\mathbf{x}-\mathbf{x}_0| - (t-t_0))} - e^{-ik(|\mathbf{x}-\mathbf{x}_0| + (t-t_0))} \right) \\
&= \frac{2\pi}{i|\mathbf{x}-\mathbf{x}_0|} 2\pi \left(\delta\left(t - t_0 - \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right) - \delta\left(t - t_0 + \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right) \right)
\end{aligned}$$

giving the advanced and retarded solutions. For clarity we have put in the speed of light explicitly.

$$\begin{aligned}
G(x^\alpha, x_0^\alpha) &= \frac{1}{4\pi^2 i} \Theta(t-t_0) \int \frac{d^3k}{\omega(k)} \, e^{i(\mathbf{k}(\mathbf{x}-\mathbf{x}_0) - \omega(t-t_0))} - \frac{1}{4\pi^2 i} \Theta(t) \int \frac{d^3k}{\omega(k)} \, e^{i(\mathbf{k}(\mathbf{x}-\mathbf{x}_0) + \omega(t-t_0))} \\
&= \frac{1}{4\pi^2 i} \Theta(t-t_0) \frac{2\pi}{i|\mathbf{x}-\mathbf{x}_0|} 2\pi \left(\delta\left(t - t_0 - \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right) - \delta\left(t - t_0 + \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right) \right) \\
&\quad - \frac{1}{4\pi^2 i} \Theta(t-t_0) \frac{2\pi}{i|\mathbf{x}-\mathbf{x}_0|} 2\pi \left(\delta\left(t - t_0 - \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right) - \delta\left(t - t_0 + \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right) \right) \\
&= -\Theta(t-t_0) \frac{1}{|\mathbf{x}-\mathbf{x}_0|} \delta\left(t - t_0 - \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right) \\
&\quad + \Theta(t-t_0) \frac{1}{|\mathbf{x}-\mathbf{x}_0|} \delta\left(t - t_0 - \frac{1}{c}|\mathbf{x}-\mathbf{x}_0| \right)
\end{aligned}$$

Something's wrong with signs and/or BC here, but let's just use the retarded part:

$$G(x^\alpha, x_0^\alpha) = \frac{\Theta(t-t_0)}{|\mathbf{x}-\mathbf{x}_0|} \delta(t - t_0 - |\mathbf{x}-\mathbf{x}_0|)$$

With this, the solution to

$$\begin{aligned}
\Box h_\pm(x) &= J_\pm(x) \\
J_\pm(x) &= \int \frac{d^3k}{2\omega} \left(A_\pm(k^i) \, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + B_\pm(k^i) \, e^{i(\mathbf{k}\cdot\mathbf{x} + \omega t)} \right)
\end{aligned}$$

is

$$\begin{aligned}
h_\pm(x) &= \int d^4x_0 G(\mathbf{x},\mathbf{x}_0,t,t_0)\, J_\pm(x_0) \\
&= \frac{1}{4\pi^2}\int d^4x_0 \int d^4k G(k)\, e^{ik_\mu\left(x^\mu - x_0^\mu\right)} J_\pm(x_0) \\
&= \int d^4k G(k)\, e^{ik_\mu x^\mu} \frac{1}{4\pi^2}\int d^4x_0 J_\pm(x_0)\, e^{-ik_\mu x_0^\mu} \\
&= \int d^4k \left(G(k)\, J_\pm(k)\right) e^{ik_\mu x^\mu}
\end{aligned}$$

This shows that

$$\begin{aligned}
\Box h_\pm(x) &= \int d^4k \left(G(k)\, J_\pm(k)\right) \Box e^{ik_\mu x^\mu} \\
&= 0
\end{aligned}$$

whenever

$$\Box\Box h_\pm(x) = 0$$

But clearly, $\Box\Box h_\pm(x) = 0$ if $\Box h_\pm(x) = 0$ so we conclude that:

$$\Box\Box h_\pm(x) = 0 \quad iff \quad \Box h_\pm(x) = 0$$

This means we have a solution to linearized Weyl gravity in vacuum if and only if it is also a solution to linearized general relativity in vacuum.